Construction and properties of fractal trees with tunable dimension: The interplay of geometry and physics

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(Received 14 May 1996)

In this paper, we emphasize three different techniques for the growth of fractal trees with a desired fractal dimension D_f. The three different growths are due to the influence of (i) stretched branches, (ii) dead ends, or (iii) a variable branching rate. Several examples are given. We point out that geometrical and physical properties (skeleton dimension, percolation exponents, self-avoiding walk) of fractal trees depend strongly on their type. The most striking result is that the critical exponents at the percolation transition are nonuniversal since they depend on the tree type. The critical exponents depend on D_f for trees of types (ii) and (iii). [S1063-651X(97)03201-7]

PACS number(s): 05.40.+j, 61.43.Hv, 68.35.Rh

I. INTRODUCTION

Loopless branched structures are often encountered in nature, such as lung, coral, starbust polymers, phylogenetic trees, etc. The kinetic growth of such branched structures with no loop is also used to describe dynamical processes such as fragmentation [1], demography [1], biological evolution [2,3], off-lattice diffusion-limited aggregation (DLA) [4,5], physiology [6], etc. Branched structures and/or processes are also used to elaborate mean-field theories such as for percolation [7,8] or for self-organized criticality [9]. In order to describe these treelike structures in which branching processes take place, self-similarity and multifractality concepts are often used [10].

The aim of the present paper is essentially to discuss both geometrical and physical properties of three different types of fractal trees, which are constructed through a growth process in high dimensions. The present work rationalizes the notion of fractal trees and emphasizes some interesting features.

II. CONSTRUCTION OF FRACTAL TREES

The most well-known tree is probably the Cayley tree [11]. This tree is generated as follows: the end of each branch is a *growth site* from which z branches of unit length grow out. Repeating indefinitely the latter growth process, this leads to the formation of a hierarchical structure as schematically drawn in Fig. 1 where the branching-growth process has been started from a single branch. The number of sites $n(g)$ in the so-called gth shell is an exponential function of z, i.e., $n(g)=z^g$.

One should remark that the notion of distance (also called the chemical distance) is only defined along the tree and cannot necessarily be connected with the Euclidian distance of the embedding space. All trees considered herein are of Cayley type; i.e., they are embedded in an infinite Euclidian space. Following this point of view, the number of sites within a distance L from the origin of a fractal tree scales as L^{D_f}. The exponent D_f is the fractal dimension of the tree and should be larger than or equal to one. Following this definition of a fractal tree, the Cayley tree has thus an asymptotically infinite dimensionality in the sense that the total number of sites within a distance L from the origin of the tree increases exponentially with L.

The skeleton of a tree is defined [12] as the set of sites belonging to the shortest paths from the root of the tree to the sites in a shell at a distance g. The important parameter characterizing such a skeleton is the fractal skeleton dimension D_f^S [12]. Skeleton substructures are important in characterizing various physical properties such as diffusion, elasticity, and resistivity.

The tree structure is characterized by three essential local parameters, which are (i) the length of the branches (i.e., the distance between two neighboring sites), (ii) the activity of the growth sites, and (iii) the branching rate z. For the usual Cayley tree, the branching rate and the length of the segments are constant parameters during the whole growth, and the branching takes place on all sites simultaneously. We will show below that by modifying each parameter during the tree growth, one can obtain a fractal tree with a desired fractal dimension D_f. The skeletons of the resulting trees will also be discussed.

(i) Stretched trees. If one deforms the Cayley tree by stretching the branches, one can obtain a self-similar tree if

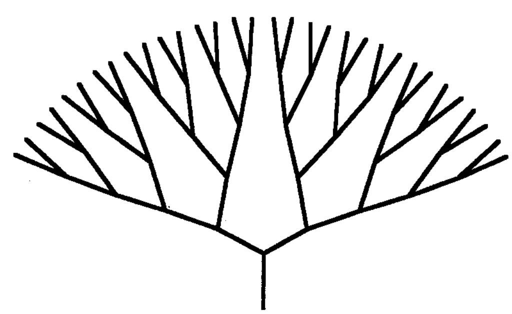

FIG. 1. The first through fifth generations of an ordinary Cayley tree with branching rate $z=2$.

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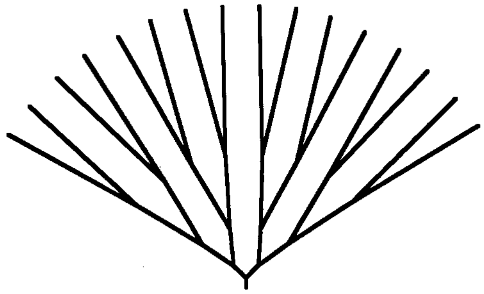

FIG. 2. The first through fourth generations of a stretched Cayley tree with $D_f=2$ and branching rate $z=2$. Fractality is obtained by stretching the length of the branches.

and only if the length of the successive segments connecting the branching points is a geometrical progression with an argument $q\geqslant 1$. Stretched Cayley trees are ordered hierarchical structures, which are useful for theoretical developments as, for example, in the study of nonanomalous diffusion on fractal objects [13]. The chemical dimension or the fractal dimension D_f of a *stretched Cayley tree* is then given by

$$D_f=1+\frac{\ln(z)}{\ln(q)}, \tag{1}$$

which ranges from 1 to $+\infty$ for any z values by tuning the stretching parameter q. Figure 2 exhibits such a stretched Cayley tree for which the length of the segments is a geometrical progression of argument $q=2$. The tree of Fig. 2 is thus fractal with a dimension $D_f=2$ since $z=2$. For $q=1$, a Cayley tree is obviously recovered. The skeleton of a stretched tree is trivially equivalent to the tree itself. Thus, the fractal dimension of the skeleton $D_f^S=D_f$ is also given by Eq. (1).

(ii) Trimmed trees. In a second type of fractal growth technique, a branching process with a constant branching rate z can be considered but sometimes imposing *dead ends*. One obtains a so-called *trimmed Cayley tree*. Cases of non-growing ''sites'' are encountered, for example, in the growth of off-lattice DLA clusters [4]. Indeed, near the origin of the DLA tree (cluster), the growth probability is extremely low due to some screening effect of the main growing branches. Other cases of non-growing sites are also encountered in evolution models of phylogeneticlike trees where the dead ends are associated to extinct species [3]. In order to generate such a tree of a given fractal dimension D_f, the number of sites $n(g)$ of the gth shell can be imposed to be a power law

$$n(g)=zg^{D_f-1}, \tag{2}$$

with an exponent D_f-1. This power law can be quite generally imposed in a stochastic manner, i.e., the distribution of nongrowing sites can be disordered on each shell. However, the ratio $n(g+1)/n(g)$ should be less than or equal to z for all g values. Thus, one has the following condition for obtaining a fractal (trimmed) tree:

$$(D_f-1)\leqslant\frac{\ln(z)}{\ln(2)}, \tag{3}$$

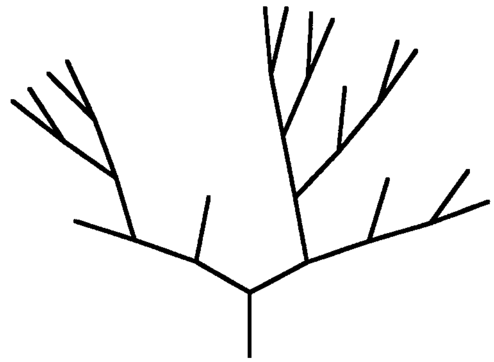

FIG. 3. The first through fifth generations of a trimmed Cayley tree with $D_f=2$ and branching rate $z=2$. Fractality is obtained by imposing a power-law distribution of sites per shell. In particular here the number of branching points on successive shells is incremented by one, giving a fractal dimension $D_f=2$.

giving an upper limit for D_f as a function of the branching ratio z. Figure 3 shows the first five stages in the construction of a trimmed Cayley tree. In Fig. 3, the number of branching points on successive shells is incremented by one, giving for this tree a dimension $D_f=2$, i.e., the upper limit dimension for a branching rate $z=2$. One should remark that a similar ''trimmed'' technique [14,15] has been previously studied for growing specific fractal trees embedded in a Euclidian space of finite dimensionality d. The fractal dimension of these trees is obviously bounded by the space dimensionality d in contrast with the present trimmed trees for which the fractal dimension is bounded by the branching rate z through Eq. (3).

In contrast with the stretched trees, the skeleton of a trimmed fractal tree is not equivalent to the tree due to the presence of dead ends. A previous analytical study [12] has shown that the dimension of the skeleton is given by

$$D_f^S=\begin{cases}1 & \text{for } D_f\leqslant 2\\ D_f-1 & \text{for } D_f\geqslant 2,\end{cases} \tag{4}$$

demonstrating the existence of a critical fractal dimension $D_f^c=2$ for fractal trimmed trees. One should remark that the existence of such a critical dimension D_f^c [15] has been also found for the skeleton of specific trees embedded in d space by Havlin *et al.*. They have found $D_f^c\approx 1.65$ in $d=2$ for the case of random values of z taken from a power law distribution [15].

(iii) Diluted trees. A third type of fractal tree consists in modifying the branching rate z from one shell to another. This third type of tree is of interest for describing natural patterns. Off-lattice DLA clusters present also a nonuniform branching rate along their tree structure [4,5]. A recent model of such a tree growth with two kinds of entities (i.e., magnetic spins) having different branching rates [16] has shown unexpected behaviors underlying the interest of this type of tree. The branching rate of such a fractal tree (the so-called *diluted Cayley tree* [16]) behaves like

$$z(g)=\left(\frac{g}{g-1}\right)^{D_f-1} \tag{5}$$

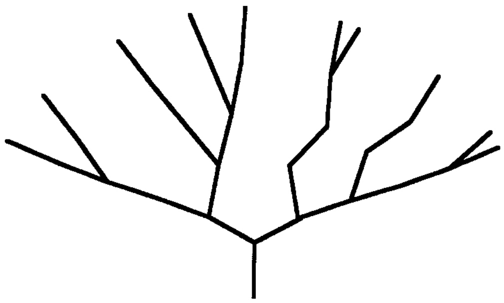

FIG. 4. The first through fifth generations of a diluted Cayley tree with $D_f = 2$ and branching rate $z = 2$. Fractality is obtained by modifying the branching rate on the various shells following Eq. (5).

for $g \geqslant 2$. Since the branching rate is locally an integer number, different integer branching rates should be used in order to satisfy Eq. (5). This can be easily made through a stochastic condition. Figure 4 shows a fractal tree with variable branching rate for $D_f = 2$. The skeleton of a diluted Cayley tree is trivially equivalent to the diluted tree itself, hence $D_f^S = D_f$.

(iv) Exotic trees. One should remark that ''exotic Cayley trees'' can be obtained by tuning a combination of the three parameters: (i) the length of the branches, (ii) the activity of growing sites, and (iii) the branching rate z. This is, for example, relevant for the study of off-lattice DLA clusters [4,5], which present both dangling ends and nonuniform branching rates along the tree. The tuning of a combination of the tree parameters is, however, outside the scope of this paper, which is essentially to present the different fundamental types of regular fractal trees (i)–(iii) and their physical properties.

One should also remark that the above trees of infinite size present self-similarity on lengths ranging from the smallest intershell distance to infinity. These fractal trees should be contrasted with fractal trees of *finite size*, which are generated by a recursive scaling procedure applied on the branches and not on the sites. The most well-known fractal tree of finite dimension is the deterministic Mandelbrot-Viscek tree [17].

III. SOME PHYSICAL PROPERTIES OF FRACTAL TREES

From what was described above, it is understood that fractal trees with the same fractal dimension can have different local structures and skeletons related to their growth condition. Thus, the physical properties are expected to be different from one type of tree to another. In order to emphasize these possible differences, we have studied two common physical phenomena: the percolation problem and the discrete self-avoiding walk (SAW) over each type of fractal tree.

A. Percolation on fractal trees

Percolation theory considers the spanning of large clusters in a disordered system, and describes how different points of a system can be connected [7,8]. For percolation problems, one considers usually particles randomly distributed on the sites of a lattice, each site of the lattice being occupied by a particle with a probability x. A phase-transition is classically observed as a function of x. Below some threshold x_c, clusters of connected particles have a finite size, while above x_c the clusters of connected particles can have an infinite size. Various physical and geometrical quantities diverge at the critical point x_c following a power law behavior allowing for the definition of critical exponents. For example, the mean cluster mass S diverges as

$$S \sim |x_c - x|^{-\gamma} \tag{6}$$

for x below x_c. For x above x_c, the average mass of the ''holes,'' i.e., the clusters of connected empty sites, diverges with the same power law. Another quantity of interest is the correlation length ξ, which can be associated to the mean size of the particle clusters below x_c, or associated to the mean size of the hole clusters above x_c. This length diverges also at the threshold x_c following a power law

$$\xi \sim |x_c - x|^{-\nu}, \tag{7}$$

with an exponent ν. In the following, we will restrict our calculations to the divergence of these two quantities (S and ξ) at x_c. Indeed, two exponents such as γ and ν are sufficient to describe static percolation critical phenomena since other ones can be derived from usual scaling relations [7].

Percolation threshold and exponents are known to be exactly solved on a few lattices only [7,8]. However, the exact solution of the percolation problem can be obtained for the Cayley tree. For such a hierarchical lattice, the percolation threshold is a function of the branching rate z (or the coordination number $z+1$) only and is trivially given by

$$x_c = 1/z. \tag{8}$$

It is also known [7] that the critical exponent values are, respectively, $\gamma = 1$ and $\nu = 1$ on an ordinary Cayley tree. One should note that the value of the exponent ν corresponds to the divergence of the correlation length measured along the tree. By that we mean that the dimensionality of the space where ξ is measured is not the Euclidian embedding space.

Since the local structure (the connectivity) of a stretched tree is equivalent to that of an ordinary Cayley tree, the values of the threshold x_c and the values of the exponents γ and ν for the stretched Cayley tree are the same as for the ordinary Cayley tree. However, the presence of nongrowing sites or the variation of the branching rate z along the tree changes locally the connectivity. A slight difference in some of the properties at the percolation threshold can thus be expected for fractal trees of types (ii) and (iii).

Let us solve the site-percolation problem on a fractal tree with dead ends. For such a trimmed Cayley tree, the probability $p(g)$ that a particle on the gth shell is connected by particles to the root of the tree can be defined recursively by

$$p(g) = p(g-1)\,\frac{n(g)}{n(g-1)}\,x, \tag{9}$$

which reduces to

TABLE I. Some geometrical and physical properties for the three different types of fractal trees.

Type of tree	Stretched	Trimmed	Diluted
Construction	Order	Disorder	Disorder
Skeleton dimension D_f^s	D_f	1 for $D_f \leqslant 2$, $D_f - 1$ for $D_f \geqslant 2$	D_f
Threshold x_c for site percolation	$1/z$	1	1
Percolation mass exponent γ	1	D_f	D_f
Percolation correlation length exponent ν	1	1	1
SAW mean free path λ_{SAW}	∞	$z\Gamma(D_f+1)/\ln(z)^{D_f+1}$	∞

$$p(g)=z\ g^{D_f-1}x^g \tag{10}$$

using Eq. (2). For g tending to infinity, this probability vanishes except for $x=x_c=1$. The value of the percolation threshold x_c is thus quite different from the values for stretched Cayley trees [Eq. (8)]. The same result is obtained for diluted Cayley trees.

The mean mass S of the particle cluster connected to the root of the tree is defined by

$$S=\sum_{g=1}^{+\infty} p(g)=z\sum_{g=1}^{+\infty} g^{D_f-1}x^g \tag{11}$$

and diverges at the critical point $x_c=1$ following the power law of Eq. (6) with an exponent $\gamma=D_f$ in the case of trimmed Cayley trees. The same exponent value $\gamma=D_f$ is obtained for the diluted Cayley trees.This result is quite original and different from that in the case of stretched Cayley trees for which the critical exponent γ is independent of D_f.

The correlation length ξ is given by

$$\xi^2=\frac{\Sigma_{g=1}^{+\infty}g^2p(g)}{\Sigma_{g=1}^{+\infty}p(g)}=\frac{\Sigma_{g=1}^{+\infty}g^{D_f+1}x^g}{\Sigma_{g=1}^{+\infty}g^{D_f-1}x^g} \tag{12}$$

using Eq. (10). It is seen that the correlation length ξ diverges at the percolation threshold $x_c=1$ following the power law of Eq. (7) with an exponent $\nu=1$; this value independent of D_f is the same for stretched and diluted Cayley trees.

Above, we have specifically solved the site-percolation problem on a fractal tree. One should remark that bond and site percolations are equivalent on such lattices. The threshold and exponent values are listed in Table I for each type of fractal tree. This table emphasizes the fact that the generation of a fractal tree is the key parameter determining percolation properties on it. Moreover, we have found that some critical exponents are a function of the fractal dimension D_f of the tree depending on the type of tree.

B. Discrete self-avoiding walk on fractal trees

This subsection is devoted to the study of a discrete SAW on a fractal tree starting from the root. On a stretched Cayley tree or on a diluted Cayley tree, no dangling end can be found such that a self-avoiding walker never stops. Thus, for such fractal trees, the mean free path λ_{SAW} diverges.

On a trimmed tree, a walker can encounter a dead end. Then, the walk is stopped there. The probability that a walker is stopped after g successive jumps is thus given by

$$\left(\frac{n(1)}{z}\right)\left(\frac{n(2)}{z\ n(1)}\right)\cdots\left(\frac{n(g)}{z\ n(g-1)}\right)\left(1-\frac{n(g+1)}{zn(g)}\right) =\{z^{-g}n(g)-z^{-(g+1)}n(g+1)\}. \tag{13}$$

The summation of $g\{z^{-g}n(g)-z^{-(g+1)}n(g+1)\}$ with g ranging from 0 to $+\infty$ gives

$$\lambda_{SAW}=z\frac{\Gamma(D_f+1)}{\ln(z)^{D_f+1}}, \tag{14}$$

where $\Gamma(x)$ is the gamma function. Thus, the mean free path λ_{SAW} is strongly dependent on z and D_f. Figure 5 presents the exact value of λ_{SAW} (obtained numerically) as a function of D_f and for various values of z. The black dots denote the upper fractal dimension D_f of these trees. One should remark that the largest mean free path is obtained for trees of dimension $D_f=2$ and branching rate $z=2$.

The different behaviors of λ_{SAW} as a function of the tree growth illustrate that other physical properties can be expected to have quite different behaviors.

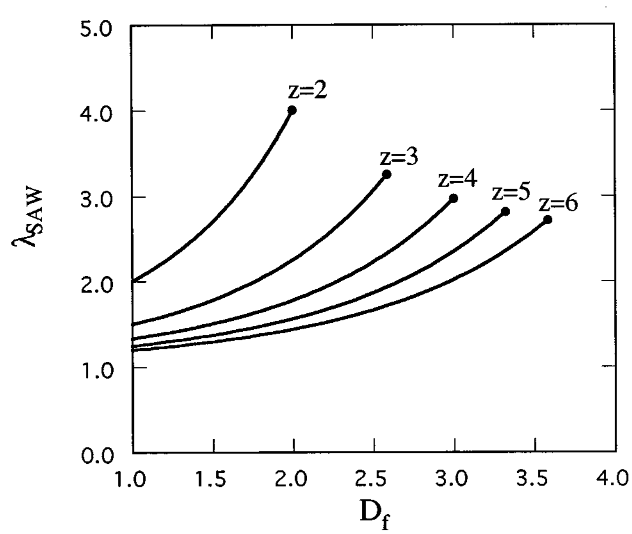

FIG. 5. The mean free path λ_{SAW} as a function of D_f for trimmed Cayley trees as in Fig. 3. Various branching rates are illustrated. The black dots denote the upper fractal dimension D_f of these trees.

IV. CONCLUSION

In summary, three different techniques for the construction of fractal trees of infinite size have been emphasized here. So-called stretched, trimmed, and diluted Cayley trees have been studied. Various examples have been given. The three different types of fractal trees have different local structures and skeletons.

We have solved exactly the percolation problem and the discrete self-avoiding walk for each tree type. The results point out that the physical properties of fractal trees depend strongly on their construction method. More importantly, we have shown that the percolation critical exponents are a function of the dimension D_f on trimmed and diluted fractal trees and are independent of this parameter for stretched trees. Moreover, a SAW on a trimmed Cayley tree is both fractal dimension D_f and branching rate z dependent.

Table I summarizes the geometrical and physical properties of the three different types of fractal trees. Our present classification of fractal trees into three classes (stretched, trimmed, and diluted) is thus relevant and rationalizes the notion of fractal trees and that of their skeleton. The mixture of the different construction techniques could also lead to fractal trees with both tunable fractal and physical properties.

ACKNOWLEDGMENTS

This work was partially supported through the ARC (94-99/174) contract of the University of Liège. N. V. thanks the Belgium Research Funds for Industry and Agriculture (FRIA, Brussels). A special grant from FNRS/LOTTO allowed us to perform specific numerical work.

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